PLANE FIELDS OF FORCE WHOSE TRAJECTORIES ARE INVARIANT UNDER A PROJECTIVE GROUP*

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A particle moving freely in a plane under the action of a force which depends only on the position of the particle has equations of motion of the form

$$m\frac{d^2x}{dt^2}=\phi(x,y),\qquad m\frac{d^2y}{dt^2}=\psi(x,y).$$

The functions $\phi(x,y)$ and $\psi(x,y)$ are assumed to be uniform and to possess first and second derivatives over the part of the plane considered. The case in which both are zero is excluded since then the field of force vanishes and the trajectories will consist only of the ∞^2 straight lines of the plane.

The particle may be projected from any position x_0, y_0, at any time t_0, and with any velocity, given by the direction y_0' and the speed v_0. By varying the four arbitrary constants x_0, y_0, y_0', v_0, all trajectories are obtained. However, since each trajectory can be described by starting from any one of its points with the proper velocity, each field of force gives rise to only a triply infinite system of curves.

The differential equation of such a triply infinite system of curves has the form†

$$y'''=Hy''^2+Gy''$$

where

$$H=\frac{-3\phi}{\psi-\phi y'},\qquad G=\frac{\psi_x+(\psi_y-\phi_x)y'-\phi_y y'^2}{\psi-\phi y'}.$$

Any differential equation of this type is transformed into another of the same type by means of the collineations of the plane.‡ There are no other contact transformations which convert every system of trajectories into a system of trajectories.§ Although the type is invariant, each single equation of this type

* Presented to the Society November 28, 1908.

† E. KASNER, *The trajectories of dynamics*, these Transactions, vol. 7 (1906), pp. 401–424.

‡ Ibid., p. 420. The importance of collineations in dynamics was first indicated by APPELL, *De l'homographie en mécanique*, American Journal of Mathematics, vol. 12 (1890).

§ E. KASNER, *Note on the transformations of dynamics* (abstract), Bulletin of the American Mathematical Society, vol. 13 (1906), p. 157.

will not in general be invariant under any collineation. The object of this paper is to determine the plane fields of force that give rise to a triply infinite system of trajectories invariant under a continuous group of projective transformations.

In sections I–VI we determine the forms of the functions $\phi(x, y)$ and $\psi(x, y)$ such that the corresponding system of trajectories is invariant under some one of the projective groups completely tabulated by LIE.* In section VII the results of the preceding sections are collected. In section VIII the fields of force corresponding to groups of three or more parameters are considered in detail; the systems of trajectories are here given in finite form, and the cases indicated in which the trajectories are conics. It is of interest to note how the familiar forces of nature appear among those whose trajectories allow the largest groups.

I. ONE-PARAMETER GROUPS.

§ 1. $xp + ayq$, $a \neq 0, 1$ [35].†

Every one-parameter continuous group of point transformations contains one and only one infinitesimal transformation

$$\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}, \qquad \text{or} \qquad \xi(x, y)p + \eta(x, y)q.$$

On the other hand each such infinitesimal transformation belongs to a single one-parameter group and consequently can be said to generate that group.‡

The increments given to x and y by such an infinitesimal transformation are

$$\delta x = \xi\delta t, \qquad \delta y = \eta\delta t.$$

The increments given to y', y'', $\cdots$, $y^{(n)}$ are determined by the relation §

$$\delta(dy^{(n-1)} - y^{(n)}dx) = 0.$$

The infinitesimal transformation $xp + ayq$, $a \neq 0, 1$, gives to x, y, y', y'', y''' the increments

$$\delta x = x\delta t, \quad \delta y = ay\delta t, \quad \delta y' = (a-1)y'\delta t, \quad \delta y'' = (a-2)y''\delta t, \quad \delta y''' = (a-3)y'''\delta t$$

We write the equation of our trajectories in the form

$$D \equiv y''' - Hy''^2 - Gy'' = 0.$$

If $D = 0$ is invariant under the group, then

$$\frac{\partial D}{\partial x}\delta x + \frac{\partial D}{\partial y}\delta y + \frac{\partial D}{\partial y'}\delta y' + \frac{\partial D}{\partial y''}\delta y'' + \frac{\partial D}{\partial y'''}\delta y''' = 0,$$

* LIE-SCHEFFERS, *Continuierliche Gruppen*, pp. 288–291.

† The number in brackets indicates the place of the corresponding group in LIE's list.

‡ By the group $\xi p + \eta q$ we shall mean the group generated by this infinitesimal transformation.

§ LIE-SCHEFFERS: *Geometrie der Berührungstransformationen*, p. 137.

by means of $D=0$ when δy, δx, etc., are replaced by their values.

Expanding this, we find

$$[xH_x+ayH_y+(a-1)y'H_{y'}+(a-1)H]y''^2+[xG_x+ayG_y+(a-1)y'G_y'+G]=0,$$

which must be satisfied identically. Hence

$$(1) \qquad xH_x+ayH_y+(a-1)H_{y'}+(a-1)H=0,$$

$$(2) \qquad xG_x+ayG_y+(a-1)G_{y'}+G=0.$$

The general solution of this set is

$$H=x^{1-a}\alpha\left(\frac{x^a}{y},\, x^{1-a}y'\right), \qquad G=\frac{1}{x}\beta\left(\frac{x^a}{y},\, x^{1-a}y'\right),$$

where α, β denote arbitrary functions of their arguments. But since H is of the form $-3\phi/(\psi-\phi y')$, we have

$$(3) \qquad \frac{-3\phi}{\psi-\phi y'}=x^{1-a}\alpha\left(\frac{x^a}{y},\, x^{1-a}y'\right)$$

Placing $x^a/y=u$, $x^{1-a}y'=v$, we find

$$(4) \qquad \frac{-3\phi}{x^{1-a}\psi-v\phi}=\alpha(u,\, v).$$

By hypothesis ϕ and ψ are functions of x and y alone. They do not contain y' and hence must be independent of v. The left-hand member of (4) can be put in the form $-3/(x^{1-a}\psi/\phi)-v$. Since this is a function of u and v, $x^{1-a}\psi/\phi$ must be a function of u alone. The most general forms of ϕ and ψ are then

$$(5) \qquad \phi=F(x,\, y)F_1\left(\frac{x^a}{y}\right),$$

$$(6) \qquad \psi=x^{a-1}F(x,\, y)F_2\left(\frac{x^a}{y}\right),$$

in which $F(x,\, y)$ is an unknown function to be determined from

$$(7) \qquad G=\frac{\psi_x+(\psi_y-\phi_x)y'-\phi_y y'^2}{\psi-\phi y'}.$$

From (5), (6), and (7), we then have*

$$(8) \quad \frac{xF_x}{F}+xy'\frac{F_y}{F}+\frac{aF_2'-\dfrac{y}{x^a}F_2+\left(F_2-\dfrac{ax^a}{y}F_1\right)x^{1-a}y'+\left(\dfrac{x^a}{y}\right)^2(x^{1-a}y')^2F_1'}{\dfrac{y}{x^a}F_2-x^{1-a}y'F_1}$$

$$=\beta\left(\frac{x^a}{y},x^{1-a}y'\right).$$

* By F_1', F_2' we shall always mean the first derivatives with respect to the single argument involved.

Since F_1, F_2, F'_1, F'_2 are functions of x^a/y alone, the third term of the left-hand member is a function of x^a/y, $x^{1-a}y'$. The other terms of the left-hand member must then be a function of the same arguments, say

$$\frac{xF_x}{F} + xy'\frac{F_y}{F} = \beta_1\left(\frac{x^a}{y}, x^{1-a}y'\right). \tag{9}$$

The function F, however, does not contain y'. Hence both xF_x/F and x^aF_y/F must be functions of x^a/y. We choose the form of these functions in the way most convenient to avoid integral signs as follows:

$$\frac{xF_x}{F} = a\frac{x^a}{y}\beta'_2\left(\frac{x^a}{y}\right), \qquad \frac{x^aF_y}{F} = \beta_3\left(\frac{x^a}{y}\right). \tag{10), (11}$$

From (10) we have

$$\log F = \beta_2\left(\frac{x^a}{y}\right) + f(y). \tag{12}$$

Hence, from (11) and (12),

$$-\frac{x^{2a}}{y^2}\beta'_2\left(\frac{x^a}{y}\right) + x^af'(y) = \beta_3\left(\frac{x^a}{y}\right).$$

We thus find

$$f'(y) = \frac{c}{y}, \qquad f(y) = \log c_1 y^c,$$

$$F(x, y) = c_1 y^c e^{\beta_2}.$$

Placing $c_1F_1e^{\beta_2} = F_3$ and $c_1F_2e^{\beta_2} = F_4$, we have as the final solution

$$\phi = y^cF_3\left(\frac{x^a}{y}\right), \qquad \psi = \frac{y^{c+1}}{x}F_4\left(\frac{x^a}{y}\right).$$

§ 2. $p + yq$ [36].

This one-parameter group gives to x, y, y', y'', y''' the infinitesimal increments

$$\delta x = \delta t, \qquad \delta y = y\delta t, \qquad \delta y' = y'\delta t, \qquad \delta y'' = y''\delta t, \qquad \delta y''' = y'''\delta t.$$

By the same course of reasoning as in the preceding case, we must have H and G such functions that they satisfy the two equations

$$H_x + yH_y + y'H_{y'} + H = 0, \tag{1}$$

$$G_x + yG_y + y'G_{y'} = 0. \tag{2}$$

Solving (1) and (2) we have

$$H = \frac{1}{y}\alpha\left(e^{-x}y, \frac{y'}{y}\right), \qquad G = \beta\left(e^{-x}y, \frac{y'}{y}\right). \tag{3), (4}$$

Therefore

$$\frac{-3\phi}{\frac{\psi}{y}-\phi\frac{y'}{y}}=\alpha\left(e^{-x}y,\frac{y'}{y}\right)$$

By a course of reasoning similar to that employed in the preceding case, we find

(5), (6) $$\phi=F(x,y).F_1(e^{-x}y),\qquad \psi=F(x,y)\cdot F_2(e^{-x}y).$$

From (4), (5), (6), and the relation connecting ϕ, ψ and G, we have

$$\frac{F_x}{F}+y'\frac{F_y}{F}+\frac{-e^{-x}yF_2'+(F_2+e^{-x}yF_2'+e^{-x}yF_1')\frac{y'}{y}-e^{-x}y\left(\frac{y'}{y}\right)^2F_1'}{F_2-\frac{y'}{y}F_1}=\beta\left(e^{-x}y,\frac{y'}{y}\right)$$

The third term of the left-hand member is a function of $e^{-x}y$, y'/y, hence F_x/F and $y(F_y/F)$ must be a function of $e^{-x}y$. We choose these unknown functions in the following convenient form:

(7), (8) $$\frac{F_x}{F}=-e^{-x}y\frac{\beta_1'(e^{-x}y)}{\beta_1(e^{-x}y)},\qquad y\frac{F_y}{F}=\beta_2(e^{-x}y).$$

Integrating (7), we have

(9) $$\log F=\log\beta_1(e^{-x}y)+f(y).$$

Then, from (8) and (9),

(10) $$e^{-x}y\frac{\beta_1'(e^{-x}y)}{\beta_1(e^{-x}y)}+yf'(y)=\beta_2(e^{-x}y),$$

$$f'(y)=\frac{c}{y},\qquad f(y)=\log c_1y^c.$$

Hence

$$F(x,y)=c_1y^c\beta_1(e^{-x}y).$$

Changing the notation, we write our result in the form

$$\phi=y^cF_3(e^{-x}y),\qquad \psi=y^{c+1}F_4(e^{-x}y).$$

§ 3. $p+xq$ [37].

The conditions to be satisfied by H and G are here

(1), (2) $$H_x+xH_y+H_{y'}=0,\qquad G_x+xG_y+G_{y'}=0.$$

The solutions are

(3), (4) $$H=\alpha(x^2-2v,x-y'),\qquad G=\beta(x^2-2y,x-y').$$

The same method of reasoning used in the two preceding cases gives

$$(5)\qquad \phi = F(x, y)\cdot F_1(x^2 - 2y),$$

$$(6)\qquad \psi = xF(x, y)\cdot F_1(x^2 - 2y) + F(x, y)\cdot F_2(x^2 - 2y).$$

From (2), (5), (6) and the relation between G and ϕ and ψ, we find

$$(7)\quad \frac{F_x}{F} + y'F + \frac{F_1 + 2F_1'\cdot(x-y')^2 + 2(x-y')F_2'}{(x-y')F_1 + F_2} = \beta(x^2 - 2y, x - y').$$

The third term of the left-hand member is a function of the two arguments $x^2 - 2y$ and $x - y'$. Then $F_x/F + y'\cdot F_y/F$ is some function of the same two arguments. We have then

$$(8), (9)\quad \frac{F_x}{F} = \frac{2x\beta_1'(x^2-2y)}{\beta_1(x^2-2y)} + \beta_2(x^2 - 2y), \qquad \frac{F_y}{F} = -\frac{2\beta_1'(x^2-2y)}{\beta_1(x^2-2y)}.$$

From (9), we have then

$$(10)\qquad \log F = \log \beta_1(x^2 - 2y) + f(x).$$

From (8) and (10), we find that

$$f'(x) = \beta_2(x^2 - 2y).$$

Hence $f'(x) = c$, $f(x) = cx + \log c_1$. Therefore

$$F(x, y) = c_1 e^{cx}\beta_1(x^2 - 2y).$$

The final solution may be written in the form

$$\phi = e^{cx}F_3(x^2 - 2y),$$

$$\psi = xe^{cx}F_3(x^2 - 2y) + e^{cx}F_4(x^2 - 2y).$$

§ 4. $xp + yq$ [38].

The result may be written

$$\phi = y^c F_3\left(\frac{x}{y}\right), \qquad \psi = y^c F_4\left(\frac{x}{y}\right).$$

These functions can be obtained from the expressions for ϕ and ψ determined for the case of the group [35] by making $a = 1$ and changing the notation.

§ 5. q [39].

The conditions that $D = 0$ shall be invariant under this, the last of the one-parameter groups, are

$$H_y = 0, \qquad G_y = 0;$$

so that

$$H = \alpha(x, y'), \qquad G = \beta(x, y').$$

Since

$$\frac{-3\phi}{\psi-\phi y'}=\alpha(x, y'),$$

we must have

$$\phi=F(x, y)\ F_1(x), \qquad \psi=F(x, y)\cdot F_2(x).$$

These give

$$\frac{F_x}{F}+y'\frac{F_y}{F}+\frac{F_2'-y'F_1'}{F_2-y'F_1}=\beta(x, y').$$

The third term of the left hand member is a function of x and y'; hence

$$\frac{F_x}{F}=\frac{\beta_1'(x)}{\beta_1(x)}, \qquad \frac{F_y}{F}=\beta_2(x).$$

From these, we find

$$F(x, y)=c_1e^{cy}\beta_1(x).$$

The resulting field is

$$\phi=e^{cy}F_3(x), \qquad \psi=e^{cy}F_4(x).$$

II. Two-parameter groups.

§ 6. $p+xq,\ q$ [24].

In order to determine the forms of ϕ and ψ such that $D=0$ shall be invariant under the group generated by two infinitesimal transformations we will first determine the form of ϕ and ψ and hence H and G such that $D=0$ is invariant under one of the infinitesimal transformations; then impose the additional conditions on H and G necessary for invariance under the second.

We have already seen (§ 3) that when $D=0$ is invariant under group [37],*

$\phi=e^{cx}F_3(x^2-2y)$ and $\psi=xe^{cx}F_3(x^2-2y)+e^{cx}F_4(x^2=2y)$ and hence

$$(1)\qquad H=\frac{-3F_3}{F_4+(x-y')F_3},$$

$$(2)\qquad G=\frac{F_3+cxF_3+xF_{3x}+F_{4x}+cF_4+(xF_{3y}+F_{4y}-cF_3-F_{3x})y'-F_{3y}y'^2}{F_4+(x-y')F_3}.$$

If $D=0$ is also invariant under q, then

$$(3), (4)\qquad H_y=0, \qquad G_y=0.$$

To be invariant under both groups, ϕ and ψ must be such functions that (1) and (2) are consistent with (3) and (4).

From (1) and (3) we have

$$(5)\qquad F_4F_{3y}-F_3F_{4y}=0,$$

* By group [37], etc., we mean throughout this paper the thirty-seventh, etc., group according to the numbering in Lie's list.

and this gives

(6) $$F_4 = f(x) \cdot F_3.$$

But F_4 and F_3 are functions of $x^2 - 2y$; hence $f(x)$ must be a constant, that is,

(7) $$F_4 = c_1 F_3.$$

From (1) and (2) we have

(8) $$G = \frac{1 + cx + cc_1 - cy'}{c_1 + x - y'} + \frac{F_{3x}}{F_3} + y' \frac{F_{3y}}{F_3}.$$

Since F_3 is independent of y' we have, from (8) and (4),

(9), (10) $$\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right) = 0, \qquad \frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = 0.$$

Therefore

(11), (12) $$\frac{F_{3x}}{F_3} = \frac{f_1'(x)}{f_1(x)}, \qquad \frac{F_{3y}}{F_3} = f_2(x).$$

From (11) we find

(13) $$\log F_3 = \log f_1(x) + f_3(y),$$

and, from (12) and (13),

$$f_3'(y) = f_2(x) = -2c_2.$$

Therefore

$$f_3(y) = -2c_2 y + \log c_3,$$

$$\log F_3 = -2c_2 y + \log f(x) + \log c_3.$$

Bnt log F_3 is a function of $x^2 - 2y$ alone. Therefore we must have

$$\log f(x) = c_2 x^2.$$

The forms of F_3, F_4 are thus found to be

$$F_3 = c_3 e^{c_2(x^2-2y)}, \qquad F_4 = c_1 c_3 e^{c_2(x^2-2y)}.$$

The corresponding field of force is

$$\phi = c_3 e^{c_2(x^2-2y)+cx}, \qquad \psi = (c_3 x + c_4) e^{c_2(x^2-2\ \ +cx} \qquad (c_4 = c_1 c_3).$$

§ 7. $p + yq$, q [25].

Using the result under group [36], and imposing the extra conditions

(1), (2) $$H_y = 0, \qquad G_y = 0$$

we find

(3) $$F_3 = c_1 e^{-x} y F_4,$$

(4 $$G = \frac{(c+1)y'}{y} + \frac{c_1 e^{-x} yy'}{y - c_1 e^{-x} yy'} + \frac{F_{4x}}{F_4} + y' \frac{F_{4y}}{F_4}.$$

Then from (4) and (2), we have

$$(5) \qquad \frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) + y' \frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) - \frac{(c+1)y'}{y^2} = 0.$$

Since F_4 is independent of y', we have

$$(6), (7) \qquad \frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) = 0, \qquad \frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) = \frac{c+1}{y^2}.$$

Then, from (7),

$$(8) \qquad \frac{F_{4x}}{F_4} = \frac{f'(x)}{f(x)}, \qquad \log F_4 = \log f(x) + f_1(y).$$

From (7) we have

$$(9) \quad \frac{F_{4y}}{F_4} = -\frac{c+1}{y} + f_2(x), \qquad \log F_4 = -(c+1)\log y + yf_2(x) + f_3(x).$$

Since the right-hand member of (8) consists of the sum of a function of x and one of y, and F_4 is itself a function of $e^{-x}y$, we must have

$$(10) \qquad f(x) = c_2 x \qquad \text{and} \qquad f_1(y) = -c_2 \log y.$$

Then from (9) and (10) we have $c_2 = c+1$. Therefore

$$F_4 = \left(\frac{e^x}{y}\right)^{c+1}, \qquad F_3 = c_1\left(\frac{e^x}{y}\right)^c.$$

The field is

$$\phi = c_1 e^{cx}, \qquad \psi = e^{(c+1)x}.$$

§ 8. $q + xp$, xq [26].

We have shown (§ 2) that when $D = 0$ is invariant under the group $p + yq$, we have $\phi = y^c F_3(e^{-x}y)$, $\psi = y^{c+1}F_4(e^{-x}y)$. The group $q + xp$ can be obtained from the $p + yq$ by the interchange of x and y. Therefore our functions must be of the form

$$\phi = x^{c+1}F_4(e^{-y}x), \qquad \psi = x^c F_3(e^{-y}x).$$

This gives

$$(1) \qquad H = \frac{-3xF_4}{F_3 - xy'F_4},$$

$$(2) \quad G = \frac{cF_3 + xF_{3x} + [(c+1)xF_4 - x^2F_{4x} + xF_{3y}]y' - x^2y'^2F_{4y}}{F_3 - xy'F_4}.$$

The additional conditions expressing invariance under xq are found to be

$$(3), (4) \qquad xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

From (1) and (3) we have

$$(5)\qquad F_3F_{4y} - F_4F_{3y} + F_4^2 = 0.$$

Let $e^{-y}x = v$. Then $F_{4y} = -v \cdot dF_4 / dv$ and likewise for F_{3y}. Then (7) becomes

$$(6)\qquad F_3F_4' - F_4F_3' - \frac{F_4^2}{v} = 0.$$

Integrating (6), we have

$$(7)\qquad F_3 = -(\log c_1 v)F_4 = -F_4 \cdot \log c_1 e^{-y}x.$$

Then, from (2) and (7), we have

$$(8)\qquad G = \frac{c}{x} + \frac{1}{xy' + \log c_1 e^{-y}x} + \frac{F_{4x}}{F_4} + y'\frac{F_{4y}}{F_4}.$$

From (4) and (8), it follows that

$$(9)\qquad x\frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) + xy'\frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) + \frac{F_{4y}}{F_4} = 0.$$

Since F_4 is independent of y', this breaks up into

$$(10), (11)\qquad \frac{\partial}{\partial y}\left(\frac{F_{4y}}{F_4}\right) = 0, \qquad x\frac{\partial}{\partial y}\left(\frac{F_{4x}}{F_4}\right) + \frac{F_{4y}}{F_4} = 0.$$

From (10), we have

$$(12)\qquad \log F_4 = yf_1(x) + f(x).$$

Since F_4 is a function of $e^{-y}x$, we must have $f(x) = c_2 \log x +$ constant, and $f_1(x) = -c_2$, so that

$$F_4 = c_3 x^{c_2} e^{-c_2 y}.$$

Substituting this expression in (11), we find $c_2 = 0$, so that

$$F_4 = c_3, \qquad F_3 = -c_3 \log c_1 e^{-y}x = +c_3 y - c_3 \log c_1 x.$$

The field is

$$\phi = c_3 x^{c+1}, \qquad \psi = c_3 y x^c - c_3 x^c \log c_1 x.$$

§ 9. $xp + ayq,\ xq$ [27].

From the first of these infinitesimal transformations we have, § 1,

$$\phi = y^c F_3\left(\frac{x^a}{y}\right), \qquad \psi = \frac{y^{c+1}}{x} F_4\left(\frac{x^a}{y}\right).$$

Here

(1) $$H = \frac{-3xF_3}{yF_4 - xy'F_3},$$

(2) $$G = \frac{\frac{y}{x}F_{4x} - \frac{y}{x^2}F_4 + \left(\frac{c+1}{x}F_4 + \frac{y}{x}F_{4y} - F_{3x}\right)y' - \left(\frac{c}{y}F_3 + F_{3y}\right)y'^2}{\frac{y}{x}F_4 - y'F_3}.$$

To be invariant under the group xq, according to § 8, H and G must satisfy the conditions

(3), (4) $$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

Equations (1) and (3) then give

(5) $$y\frac{F_3F_{4y} - F_4F_{3y}}{F_3^2} + \frac{F_4}{F_3} - 1 = 0.$$

Integrating, we find

(6), (7) $$y\frac{F_4}{F_3} = y + f(x), \qquad F_4 = \left(1 + \frac{f(x)}{y}\right)F_3.$$

Since F_4 and F_3 are functions of x^a/y, $[1 + f(x)/y]$ must be a function of the same argument. Therefore

(8) $$f(x) = c_1x^a, \qquad F_4 = \left(1 + \frac{c_1x^a}{y}\right)F_3.$$

From (8) and (2), together with (4), we have

(9) $$\frac{c}{y} - \frac{cxy'}{y^2} + \frac{F_{3y}}{F_3} + x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right) + xy'\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = 0.$$

Since F_3 is independent of y',

(10), (11) $$\frac{c}{y} + \frac{F_{3y}}{F_3} + x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right) = 0, \qquad x\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) - \frac{cx}{y^2} = 0.$$

Putting $x^a/y = v$, we have

$$F_{3y} = -\frac{x^a}{y^2}\frac{dF_3}{dv} = -\frac{x^a}{y^2}F'_3,$$

$$\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = \frac{\partial}{\partial y}\left(-\frac{x^a}{y^2}\frac{F'_3}{F_3}\right) = \left(\frac{x^a}{y^2}\right)^2\frac{d}{dv}\left(\frac{F'_3}{F_3}\right) + \frac{2x^a}{y^3}\frac{F'_3}{F_3}.$$

Then (10) becomes

(12) $$v^2\frac{d}{dv}\left(\frac{F'_3}{F_3}\right) + 2v\frac{F'_3}{F_3} = c.$$

Therefore

$$\frac{F_3'}{F_3} = \frac{c}{v} + \frac{c_2}{v_2},$$

(13)
$$F_3 = c_3 \left(\frac{x^a}{y} \right)^c e^{-\frac{c_2 y}{x^a}}.$$

From (13) and (10), we have $c_2(a-1) = 0$ and hence $c_2 = 0$, since $a \neq 1$ by hypothesis. Therefore

$$F_3 = c_3 \left(\frac{x^a}{y} \right)^c, \qquad F_4 = c_3 \left(\frac{x^a}{y} \right)^c + c_1 c_3 \left(\frac{x^a}{y} \right)^{c+1}$$

The field is

$$\phi = c_3 x^{ac}, \qquad \psi = c_3 y x^{ac-1} + c_4 x^{ac+a-1}.$$

§ 10. xp, q [28].

The conditions that $D = 0$ shall be invariant under xp are

$$xH_x - y'H_{y'} - H = 0, \qquad xG_x - y'G_{y'} + G = 0.$$

Omitting the discussion, we give the result

$$\phi = x^c F_3(y), \qquad \psi = x^{c-1} F_4(y).$$

Then we have

$$H = \frac{-3xF_3}{F_4 - xy'F_3},$$

$$G = \frac{\frac{c-1}{x} F_4 + (F_4' - cF_3)y' - xy'^2 F_3'}{F_4 - xy'F_3}.$$

From the second generator, q, we have

$$H_y = 0, \qquad G_y = 0.$$

$$F_3' F_4 - F_4' F_3 = 0, \qquad F_3 = c_1 F_4.$$

The forms of the F functions are found to be

$$F_4 = c_3 e^{c_2 y}, \qquad F_3 = c_3 c_1 e^{c_2 y};$$

and the corresponding field is

$$\phi = c_3 x^c e^{c_2 y}, \qquad \psi = c_4 x^{c-1} e^{c_2 y}.$$

§ 11. p, q [29].

When $D = 0$ is invariant under the group q (§ 5), we have

$$\phi = e^{cy} F_3(x), \qquad \psi = e^{cy} F_4(x).$$

The infinitesimal transformations p and q are equivalent since the transformation $x_1 = y$, $y_1 = x$ changes one into the other. Hence we must have also

$$\phi = e^{cx} F_5(y), \qquad \psi = e^{c} F_6(y).$$

The final result is the field

$$\phi = c_3 e^{cx + c_2 y}, \qquad \psi = c_4 e^{cx + c_2 y}.$$

§ 12. q, xq [30].

From § 5, we have $\phi = e^{cy} F_3(x)$ and $\psi = e^{cy} F_4(x)$. Also

(1), (2) $$H = \frac{-3F_3}{F_4 - y' F_3}, \qquad G = cy' + \frac{F'_4 - y' F'_3}{F_4 - y' F_3}.$$

That this may be invariant under the group xq, we must have, (§ 8),

(3), (4) $$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

From (1) and (3) we have $F_3 = 0$, and from (2), (4) and (5) we have $c = 0$. Therefore

$$\phi = 0, \qquad \psi = F_4(x).$$

§ 13. xp, yq [31].

From § 10, we have

(1), (2) $$H = \frac{-3xF_3}{F_4 - xy' F_3}, \qquad G = \frac{\frac{c-1}{x} F_4 + (F'_4 - cF_3) y' - xy'^2 F'_3}{F_4 - xy' F_3}.$$

The extra conditions, from yq, are

(3), (4) $$yH_y + y' H_{y'} + H = 0, \qquad yG_y + y' G_{y'} = 0.$$

From (1) and (3)

$$y(F'_3 F_4 - F'_4 F_3) + F_3 F_4 = 0.$$

Integrating this we have

(5) $$F_4 = c_1 y F_3.$$

From (2), (4) and (5)

$$yF''_3 F_3 - yF'^2_3 + F_3 F'_3 = 0.$$

Integration gives

$$F_3 = c_3 y^{c_2}.$$

The result is

$$\phi = c_3 x^c y^{c_2}, \qquad \psi = c_4 x^{c-1} y^{c_2+1}$$

§ 14. $xp + yq$, q [32].

From § 4, we have

$$\phi = y^c F_3\left(\frac{x}{y}\right), \qquad \psi = y^c F_4\left(\frac{x}{y}\right).$$

Also

(1), (2) $$H=\frac{-3F_3}{F_4-y'F_3},\qquad G=\frac{cy'}{y}+\frac{F_{4x}+(F_{4y}-F_{3x})y'-F_{3y}y'^2}{F_4-y'F_3}.$$

The additional requirements are

(3), (4) $$H_y=0,\qquad G_y=0.$$

Then (1) and (3) give

$$F_4\frac{\partial}{\partial y}F_3-F_3\frac{\partial}{\partial y}F_4=0.$$

But F_3 and F_4 are both functions of x/y. Therefore

(5) $$F_4=c_1F_3.$$

From (2), (4) and (5) we have

$$-\frac{cy'}{y^2}+\frac{F_{3xy}F_3-F_{3x}F_{3y}}{F_3^2}+y'\frac{F_{3yy}F_3-F_{3y}^2}{F_3^2}=0.$$

Since F_3 is independent of y', we have

(6), (7) $$\frac{F_{3xy}F_3-F_{3x}F_{3y}}{F_3^2}=0,\qquad -\frac{c}{y}+\frac{F_{3yy}F_3-F_{3y}^2}{F_3^2}=0;$$

and thus

(8), (9) $$\frac{F_{3y}}{F_3}=f(y),\qquad \frac{c}{y}+\frac{F_{3y}}{F_3}=f_1(x).$$

From these,

$$f(y)+\frac{c}{y}=f_1(x),$$

(10) $$f_1(x)=c_2,\qquad f(y)=c_2-\frac{c}{y}.$$

Then from (8) and (10) we have

$$\log F_3=c_2y-c\log y+f_2(x),\qquad F_3=\frac{1}{y^c}e^{c_2y+f_2(x)}.$$

But F_3 is a function of x/y and hence $f_2(x)$ must be such a function that the right-hand member is also a function of x/y. Therefore

$$f_2(x)=c\log x,\qquad c_2=0,$$

$$F_3=\left(\frac{x}{y}\right)^c,\qquad F_4=c_1\left(\frac{x}{y}\right).$$

The force is found to be

$$\phi=x^c,\qquad \psi=c_1x^c.$$

§ 15. q, yq [33].

From § 5 we have $\phi = e^{cy}F_3(x)$, $\psi = e^{cy}F_4(x)$, and thus

(1), (2) $$H = \frac{-3F_3}{F_4 - y'F_3}, \qquad G = cy' + \frac{F'_4 - F'_3 y'}{F_4 - F_3 y'}.$$

Invariance under the operator yq requires the two conditions

(3), (4) $$yH_y + y'H_{y'} + H = 0, \qquad yG_y + y'G_{y'} = 0.$$

From (1) and (3), we have $F_3F_4 = 0$; therefore $F_3 = 0$ or $F_4 = 0$, since we have excluded the case in which both ϕ and ψ vanish. If $F_3 = 0$, from (2) and (4) we have $c = 0$. If $F_4 = 0$, (2) and (4) again give $c = 0$. Therefore

$$\phi = 0, \ \psi = F_4(x) \qquad \text{or} \qquad \phi = F_3(x), \ \psi = 0.$$

§ 16. $xp + 2yq$, $p + xq$ [34].

When $D = 0$ is invariant under group [35], we have (§ 1)

$$\phi = y^c F_3\left(\frac{x^a}{y}\right), \qquad \psi = \frac{y^{c+1}}{x} F_4\left(\frac{x^a}{y}\right).$$

Therefore to make it invariant under the group $xp + 2yq$, we must have

$$\phi = y^c F_3\left(\frac{x^2}{y}\right), \qquad \psi = \frac{y^{c+1}}{x} F_4\left(\frac{x^2}{y}\right).$$

We may put ψ in the form

$$\psi = y^c x F_5\left(\frac{x^2}{y}\right).$$

Then

(1) $$H = \frac{-3F_3}{xF_5 - y'F_3},$$

(2) $$G = \frac{cy'}{y} + \frac{yF_5 + xyF_{5x} + (xyF_{5y} - yF_{3x})y' - yy'^2 F_{3y}}{xyF_5 - yy'F_3}.$$

In order to have $D = 0$ invariant under the group [34] we must adjoin the two conditions (§ 3)

(3), (4) $$H_x + xH_y + H_{y'} = 0, \qquad G_x + yG_y + G_{y'} = 0.$$

From (1) and (3), we have

(5) $$x(F_{3x}F_5 - F_3F_{5x}) + x^2(F_{3y}F_5 - F_3F_{5y}) + F_3^2 - F_3F_5 = 0.$$

If we put $x^2/y = v$, (5) becomes

$$(2v - v^2)(F'_3F_5 - F'_5F_3) + F_3^2 - F_3F_5 = 0.$$

Integrating, we have

$$(6) \qquad F_5 = \left(1 + \frac{c_1}{x}\sqrt{x^2 - 2y}\right) F_3.$$

From (2) and (6), we have

$$(7) \qquad G = \frac{cy'}{y} + \frac{c_1 x - c_1 y' + \sqrt{x^2 - 2y}}{(x - y' + c_1\sqrt{x^2 - 2y})\sqrt{x^2 - 2y}} + \frac{F_{3x}}{F_3} + y'\frac{F_{3y}}{F_3}.$$

From (4) and (7), we have

$$\frac{c}{y} - \frac{cxy'}{y^2} + \frac{\partial}{\partial x}\left(\frac{F_{3x}}{F_3} + y'\frac{F_{3y}}{F_3}\right) + x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3} + y'\frac{F_{3y}}{F_3}\right) + \frac{F_{3y}}{F_3} = 0.$$

Since F_3 is independent of y'

$$(8) \qquad \frac{c}{y} + \frac{\partial}{\partial x}\left(\frac{F_{3x}}{F_3}\right) + x\frac{\partial}{\partial y}\left(\frac{F_{3x}}{F_3}\right) + \frac{F_{3y}}{F_3} = 0,$$

$$(9) \qquad -\frac{cx}{y^2} + \frac{\partial}{\partial x}\left(\frac{F_{3y}}{F_3}\right) + x\frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = 0.$$

Again let $x^2/y = v$. Then

$$\frac{\partial}{\partial x}\left(\frac{F_{3y}}{F_3}\right) = -\frac{2x}{y^2}\frac{F_3'}{F_3} - \frac{2x^3}{y^3}\frac{d}{dv}\left(\frac{F_3'}{F_3}\right); \qquad \frac{\partial}{\partial y}\left(\frac{F_{3y}}{F_3}\right) = \frac{2x^2}{y^3}\frac{F_3'}{F_3} - \frac{x^4}{y^4}\frac{d}{dv}\left(\frac{F_3'}{F_3}\right).$$

Hence (9) can be written

$$-\frac{cx}{y^2} + 2\left(\frac{x^3}{y^3} - \frac{x}{y^2}\right)\frac{F_3'}{F_3} + \left(\frac{x^5}{y^4} - \frac{2x^3}{y^3}\right)\frac{d}{dv}\left(\frac{F_3'}{F_3}\right) = 0.$$

Dividing by x/y^2 and replacing x^2/y by v we have

$$(10) \qquad c - 2(v-1)\frac{F_3'}{F_3} - (v^2 - 2v)\frac{d}{dv}\left(\frac{F_3'}{F_3}\right) = 0.$$

Integrating (10), we have

$$\log F_3 = c\log(v-2) + \frac{c_2}{2}\log\frac{v-2}{v} + \log c_3,$$

$$F_3 = c_3\left(\frac{x^2 - 2y}{y}\right)^c\left(\frac{x^2 - 2y}{x^2}\right)^{\frac{c_2}{2}}.$$

Then, from (8), we have $c_2 = 0$; accordingly we have

$$F_3 = c_3\left(\frac{x^2 - 2y}{y}\right)^c, \qquad F_5 = c_3\left(\frac{x^2 - 2y}{y}\right)^c + \frac{c_1 c_3}{x}\left(\frac{x^2 - 2y}{y^c}\right)^{c+\frac{1}{2}}.$$

The field is found to be

$$\phi = c_3(x^2 - 2y)^c, \qquad \psi = (x^2 - 2y)^c(c_3 x + c_4\sqrt{x^2 - 2y}).$$

III. Three-parameter groups.

§ 17. xq, q, p [13].

We proceed from two-parameter to three-parameter groups in the same way that we proceeded from one-parameter to two-parameter groups. We choose the ϕ and ψ corresponding to a two-parameter subgroup, determine the form of H and G, and then impose the condition for invariance under a third infinitesimal transformation.

It was found, § 12, that when $D = 0$ is invariant under group [30] we have $\phi = 0$ and $\psi = F_4(x)$, and hence

(1), (2) $$H = 0, \qquad G = \frac{F'_4}{F_4}.$$

To the two-parameter group we adjoin the generator p making the group [13]. The extra conditions are

(3), (4) $$H_x = 0, \qquad G_x = 0.$$

Then from (2) and (4) we have $F_4 = e^{cx}$. The resulting field is

$$\phi = 0, \qquad \psi = e^{cx}.$$

§ 18. $p + xq, xp + 2yq, q$ [14].

When $D = 0$ is invariant under group [34] we have (§ 16) $\phi = c_3(x^2 - 2y)^c$, $\psi = (x^2 - 2y)^c(c_3 x + c_4\sqrt{x^2 - 2y})$. Here

(1) $$H = \frac{-3c_3}{c_3(x - y') + c_4\sqrt{x^2 - 2y}},$$

(2) $$G = \frac{c_4(x - y') + c_3\sqrt{x^2 - 2y}}{[c_3(x - y') + c_4\sqrt{x^2 - 2y}]\sqrt{x^2 - 2y}} + \frac{2c(x - y')}{x^2 - 2y}.$$

To render $D = 0$ invariant under the group q we must have in addition (§ 5),

(3), (4) $$H_y = 0, \qquad G_y = 0.$$

Then from (1) and (3) we have $c_3 c_4 = 0$. If $c_3 = 0$, (2) and (4) give $c = -\frac{1}{2}$. If $c_4 = 0$, (2) and (4) give $c = 0$. The field is therefore of one of the two forms

$$\phi = 0, \quad \psi = c_4; \qquad \phi = c_3, \quad \psi = c_3 x.$$

§ 19. $p + yq, q, xq$ [15].

We have found (§ 7) that when $D = 0$ is invariant under group [25] $\phi = c_1 e^{cx}$, $\psi = e^{(c+1)x}$. Then

(1), (2) $$H = \frac{-3c_1}{e^x - c_1 y'}, \qquad G = \frac{(c + 1)e^x - cc_1 y'}{e^x - c_1 y'}.$$

For invariance under the group xq, we have (§ 8) the conditions

$$(3), (4) \qquad xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

Then from (1) and (3) we have $c_1 = 0$; (2) and (4) are then consistent. Therefore

$$\phi = 0, \qquad \psi = e^{(c-1)x}.$$

§ 20. $p, q, xp + (y - x)q$ [16].

When $D = 0$ is invariant under group [29] we have (§ 11) $\phi = c_3 e^{cx+c_2y}$, $\psi = c_4 e^{cx+c_2y}$. Then

$$(1), (2) \qquad H = \frac{-3c_3}{c_4 - c_3 y'}, \qquad G = c + c_2 y'.$$

From the third generator of our group we find

$$(3) \qquad xH_x + (y - x)H_y - H_{y'} = 0,$$

$$(4) \qquad xG_x + (y - x)G_y - G_{y'} + G = 0.$$

From (2) and (4) we have $c - c_2 + c_2 y' = 0$, whence $c = c_2 = 0$. From (1) and (3) we have then $c_3 = 0$. The field is thus simply

$$\phi = 0, \qquad \psi = c_4.$$

§ 21. $xq, xp - yq, yp$ [17].

From § 9, $\phi = c_3 x^{ac}$, $\psi = c_3 y x^{ac-1} + c_4 x^{ac+a-1}$. If $a = -1$ group [27] becomes the group $xq, xp - yq$. Then we have

$$\phi = \frac{c_3}{x^c}, \qquad \psi = \frac{c_3 y}{x^{c+1}} + \frac{c_4}{x^{c+2}},$$

and

$$(1), (2) \quad H = \frac{-3c_3 x^2}{c_3 xy + c_4 - c_3 x^2 y'}, \qquad G = \frac{-(c+2)c_4}{c_3 xy + c_4 - c_3 x^2 y'} - \frac{c+1}{x}.$$

For invariance under the group yp we must have

$$(3) \qquad yH_x - y'^2 H_{y'} - 2y'H + 3 = 0,$$

$$(4) \qquad yG_x - y'^2 G_{y'} + y'G = 0.$$

From (1) and (3) we find $c_4 = 0$. Then from (2), (4) and (5) we have $c = -1$. The field is

$$\phi = c_3 x, \qquad \psi = c_3 y.$$

§ 22. $p, q, (a-1)xp + ayq$ [18].

From group [29] we have $\phi = c_3 e^{cx+c_2y}$, $\psi = c_4 e^{cx+c_2y}$ (§ 11), and thus

(1), (2) $$H = \frac{-3c_3}{c_4 - c_3 y'}, \qquad G = c + c_2 y'.$$

The further conditions are

(3) $$(a-1)xH_x + ayH_y + y'H_{y'} + H = 0,$$

(4) $$(a-1)xG_x + ayG_y + y'G_{y'} + (a-1)G = 0.$$

Then from (2) and (4) we have

$$y'c_2 + (a-1)c + (a-1)c_2 y' = 0.$$

Therefore $c = 0$, $c_2 = 0$ when $a \neq 0$; $c = 0$ when $a = 0$; $c_2 = 0$ when $a = 1$. From (1) and (3) we have $c_3 c_4 = 0$; so that $c_3 = 0$ or $c_4 = 0$. The results are

$$\left.\begin{array}{ll} \phi = 0, & \psi = c_4 \\ \phi = c_3, & \psi = 0 \end{array}\right\} \text{ when } a \neq 0, 1;$$

$$\left.\begin{array}{ll} \phi = 0, & \psi = c_4 e^{c_2 y} \\ \phi = c_3 e^{c_2 y}, & \psi = 0 \end{array}\right\} \text{ when } a = 0;$$

$$\left.\begin{array}{ll} \phi = 0, & \psi = c_4 e^{cx} \\ \phi = c_3 e^{cx}, & \psi = 0 \end{array}\right\} \text{ when } a = 1.$$

§ 23. $xp + ayq, xq, q$ [19].

From group [27], $\phi = c_3 x^{ac}$, $\psi = c_3 y x^{ac-1} + c_4 x^{ac+a-1}$, and

(1) $$H = \frac{-3c_3 x}{c_3 y + c_4 x^a - c_3 x y'},$$

(2) $$G = \frac{c_4 a x^a}{c_3 xy + c_4 x^{a+1} - c_3 x^2 y'} + \frac{ac-1}{x}.$$

For invariance under the group q, we must have (§ 5)

(3), (4) $$H_y = 0, \qquad G_y = 0.$$

From (1) and (3) we find $c_3 = 0$. Therefore

$$\phi = 0, \qquad \psi = c_4 x^{ac+a-1}.$$

§ 24. yq, xp, q [20].

From the result for group [31], we have, § 13,

(1) $$H = \frac{-3c_3 x}{c_4 y - c_3 x y'},$$

(2) $$G = \frac{c}{x} + \frac{c_2 y'}{y} + \frac{c_4(xy' - y)}{c_4 xy - c_3 x^2 y'}.$$

The further conditions are

$$H_y = 0, \qquad G_y = 0.$$

The resulting fields are found to be

$$\phi = c_3 x^c, \ \psi = 0; \qquad \phi = 0, \ \psi = c_4 x^{c-1}.$$

§ 25. $xp + yq, q, p$ [21].

From group [32] we have (§ 14) $\phi = x^c$, $\psi = c_1 x^c$, and thus

$$H = \frac{-3}{c_1 - y'}, \qquad G = \frac{c}{x}.$$

The extra conditions give $c = 0$; hence $\phi = 1$, $\psi = c_1$.

§ 26. q, xq, yq [22].

From group [30] we have, § 12, $\phi = 0$, $\psi = F_4(x)$, and thus

$$H = 0, \qquad G = \frac{F_4'}{F_4}.$$

The third operator, yq, imposes no extra restrictions. Therefore

$$\phi = 0, \qquad \psi = F_4(x).$$

§ 27. $p + xq, xp + 2yq, (x^2 - y)p + xyq$ [23].

From the group [34], we have

$$\phi = c_3(x^2 - 2y)^c, \qquad \psi = (c_3 x + c_4 \sqrt{x^2 - 2y})(x^2 - 2y)^c,$$

and hence

$$H = \frac{-3c_3}{c_3(x - y') + c_4\sqrt{x^2 - 2y}}, \tag{1}$$

$$G = \frac{c_4(x - y') + c_3\sqrt{x^2 - 2y}}{(c_3(x - y') + c_4\sqrt{x^2 - 2y})\sqrt{x^2 - 2y}} + \frac{2c(x - y')}{x^2 - 2y}. \tag{2}$$

The additional equations are

$$(x^2 - y)H_x + xyH_y + (y - xy' + y'^2)H_{y'} + 2y'H - xH - 3 = 0, \tag{3}$$

$$(x^2 - y)G_x + xyG_y + (y - xy' + y'^2)G_{y'} + 2xG - y'G + 3 = 0. \tag{4}$$

From (1) and (3) we have $c_3 = c_4 = 0$. There is then no field of force whose trajectories are invariant under the group [23].

IV. Four-parameter groups.

§ 28. $p, q, xq, axp + yq, a \neq \frac{1}{2}$ [7].

The system of trajectories is invariant under group [13] when we have (§ 17) $\phi = 0$, $\psi = e^{cx}$, and thus

(1), (2) $$H = 0, \qquad G = c.$$

The further conditions, due to the infinitesimal transformation $axp + yq$, are

(3) $$axH_x + yH_y + (1-a)y'H_{y'} + (1-a)H = 0,$$

(4) $$axG_x + yG_y + (1-a)y'G_{y'} + aG = 0.$$

From (2) and (4), we find $ac = 0$. The results are

$$\phi = 0, \ \psi = 1, \ a \neq 0; \qquad \phi = 0, \ \psi = e^{cx}, \ a = 0.$$

§ 29. $q, xq, xp + 2yq, p$ [8].

When $a = 2$ group [19] becomes the three-parameter group $q, xq, xp + 2yq$. In this case we have $\phi = 0$, $\psi = c_4 x^{2c+1}$, and

$$H = 0, \qquad G = \frac{2c+1}{x}.$$

Invariance under the operator p is easily found to require $2c + 1 = 0$. Therefore the field is

$$\phi = 0, \qquad \psi = c_4.$$

§ 30. p, q, xq, xp [9].

From group [13] we have (§ 17) $\phi = 0$, $\psi = e^{cx}$, and

$$H = 0, \qquad G = c.$$

Invariance under xp requires the conditions

$$xH_x - y'H_{y'} - H = 0, \qquad xG_x - y'G_{y'} + G = 0.$$

We find $c = 0$ and thus

$$\phi = 0, \qquad \psi = 1.$$

§ 31. xp, yq, xq, yp [10].

In order to determine ϕ and ψ so that $D = 0$ is invariant under group [10] we take the values of ϕ and ψ corresponding to the group [31], impose first the condition for invariance under the group xq, and then the condition for invariance under the group yp.

From group [31] we have (§ 13) $\phi = c_3 x^c y^{c_2}$, $\psi = c_4 x^{c-1} y^{c_2+1}$, and hence

(1), (2) $$H = \frac{-3c_3 x}{c_4 y - c_3 xy'}, \qquad G = \frac{c}{x} + \frac{c_2 y'}{y} + \frac{c_4(xy' - y)}{c_4 xy - c_3 x^2 y'}.$$

The operator xq imposes the conditions (§ 8)

$$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0. \tag{(3), (4)}$$

From (1) and (3) we have $c_3(c_3 - c_4) = 0$, so that $c_3 = 0$ or $c_3 = c_4$. If $c_3 = 0$, (2) and (4) give $c_2 = -1$. If $c_3 = c_4$, then (2) and (4) give $c_2 = 0$. Therefore

$$\phi = 0, \qquad \psi = c_4 x^{c-1}, \tag{5}$$

$$\phi = c_3 x^c, \qquad \psi = c_3 x^{c-1} y. \tag{6}$$

We must now add the conditions due to yp (§ 21), namely,

$$yH_x - y'^2 H_{y'} - 2y'H + 3 = 0, \qquad yG_x + y'^2 G_{y'} + y'G = 0. \tag{(7), (8)}$$

When ϕ and ψ are defined by (5) we have

$$H = 0, \qquad G = \frac{c-1}{x} \tag{(9), (10)}$$

and when ϕ and ψ are defined by (6) we have

$$H = \frac{-3x}{y - xy'}, \qquad G = \frac{c-1}{x}. \tag{(11), (12)}$$

Equations (7) and (9) are not consistent, hence (5) does not give a solution. Equations (7) and (11) are consistent, and (8) and (12) give $c = 1$. Therefore

$$\phi = c_3 x, \qquad \psi = c_3 y.$$

§ 32. xp, yq, q, p [11].

From group [20] (§ 24) either $\phi = c_3 x^c$, $\psi = 0$ or $\phi = 0$, $\psi = c_4 x^{c-1}$ and hence

$$H = \frac{3}{y'}, \qquad G = \frac{c}{x}, \qquad \text{or} \qquad H = 0, \qquad G = \frac{c-1}{x}.$$

The operator p requires $H_x = 0$, $G_x = 0$; from which we find

$$\phi = c_3, \quad \psi = 0; \qquad \phi = 0, \quad \psi = c_4.$$

§ 33. q, yq; xp, xq [12].

Under group [20], formulas (1), (2), (3), (4) (§ 32), give us the forms of H and G. The additional equations, due to xp, are

$$xH_y + H_{y'} = 0, \qquad xG_y + G_{y'} = 0.$$

The only solution is found to be

$$\phi = 0, \qquad \psi = c_4 x^{c-1}.$$

V. Five-parameter groups.

§ 34. $xp - yq$, yp, xq, p, q [4].

Under group [17], we have (§ 21) $\phi = c_3 x$, $\psi = c_3 y$, and hence

$$H = \frac{-3x}{y - xy'}, \qquad G = 0.$$

The operator p requires

$$H_x = 0, \qquad G_x = 0.$$

The conditions are not consistent, hence there is no field of force whose trajectories are invariant under group [4].

§ 35. $xp - yq$, yp, xq, $x^2p + xyq$, $xyp + y^2q$ [5].

Under group [17], we have (§ 34)

$$H = \frac{-3x}{y - xy'}, \qquad G = 0.$$

The operator $x^2p + xyq$ gives

$$x^2 H_x + xyH_y + (y - xy')H_{y'} - xH = 0,$$

$$x^2 G_x + xyG_y + (y - xy')G_{y'} + 2xG + 3 = 0.$$

Hence no field of force exists.

§ 36. q, yq, xq, xp, p [6].

Under group [12], $\phi = 0$, $\psi = c_4 x^{c-1}$, and hence

$$H = 0, \qquad G = \frac{c-1}{x}.$$

The operator p requires that $c = 1$. Hence the field is

$$\phi = 0, \qquad \psi = c_4.$$

VI. Groups of more than five parameters.

The remaining groups are two of six parameters

$$p, q, xp, yp, xq, yq \ [2],$$

$$xp, yp, xq, yq, x^2p + xyq, xyp + y^2q \ [3],$$

and the entire eight-parameter group

$$p, q, xp, yp, xq, yq, x^2p + xyq, xyp + y^2q \ [1].$$

Each of these contains as a subgroup either group [4] or group [5], which do not give rise to fields. Hence no field of force* corresponds to the present groups.

* Except, of course, the trivial field $\phi = 0$, $\psi = 0$, whose trajectories are straight lines.

VII. Table of the fields obtained.

The preceding results, simplified by projective reduction, and arranged in the order of LIE's list, are collected in the following table. The groups omitted, namely, 1, 2, 3, 4, 5, and 23, do not give rise to any solutions.

LIE's No.	Group.		ϕ	ψ
6	q, yq, xq, xp, p		0	1
7	$p, q, xq, axp+yq, a \neq \frac{1}{2}$	$a \neq 0$	0	1
		$a = 0$	0	e^x
8	$q, xq, xp+2yq, p$		0	1
9	p, q, xq, xp		0	1
10	xp, yq, xq, yp		x	y
11	xp, yq, q, p		0	1
12	q, yq, xp, xq		0	x^c
13	xq, q, p		0	e^x
14	$p+xq, xp+2yq, q$		0	1
			1	x
15	$p+yq, q, xq$		0	e^x
16	$p, q, xp+(y-x)q$		0	1
17	$xq, xp-yq, yp$		x	y
18	$p, q, (a-1)xp+ayq$	$a \neq 0, 1$	1	0
			0	1
		$a=0$	0	e^y
			e^y	0
		$a=1$	0	e^x
			e^x	0
19	$xp+ayq, xq, q$		0	x^{ac}
20	yq, xp, q		x^c	0
			0	x^c
21	$xp+yq, q, p$		1	1
22	q, xq, yq		0	$F(x)$
24	$p+xq, q$		$e^{c_2(x^2-2y)+cx}$	$(c_1 x+1)e^{c_2(x^2+2y)+cx}$
25	$p+yq, q$		e^{cx}	$e^{(c+1)x}$
26	$q+xp, xq$		x^{c+1}	$x^c(y-\log c_1 x)$
27	$xp+ayq, xq$		x^{ac}	$x^{ac-1}(y+c_1 x^a)$
28	xp, q		$x^{c_1} e^{cy}$	$x^{c_1-1} e^{cy}$
29	p, q		$e^{cx+c_1 y}$	$c_3 e^{cx+c_1 y}$
30	q, xq		0	$F(x)$
31	xp, yq		$x^c y^{c_1}$	$c_\lambda x^{c-1} y^{c_1+1}$

Lie's No.	Group.	ϕ	ψ
32	$xp + yq, q$	x^c	$c_1 x^c$
33	q, yq	0 $F_1(x)$	$F(x)$ 0
34	$xp + 2yq, p + xq$	$(x^2 - 2y)^c$	$(x^2-2y)^c(c_1\sqrt{x^2-2y}+x)$
35	$xp + ayq \quad a \neq 0, 1$	$y^c F\left(\frac{x^a}{y}\right)$	$\frac{y^{c+1}}{x} F_1\left(\frac{x^a}{y}\right)$
36	$p + yq$	$y^c F(e^{-x}y)$	$y^{c+1}F_1(e^{-x}y)$
37	$p + xq$	$e^{cx}F(x^2-2y)$	$xe^{cx}F(x^2-2y) + e^{cx}F_1(x^2-2y)$
38	$xp + yq$	$y^c F\left(\frac{x}{y}\right)$	$y^c F_1\left(\frac{x}{y}\right)$
39	q	$e^{cy}F(x)$	$e^{cy}F_1(x)$

VIII. Trajectories of the fields of force corresponding to groups of three or more parameters.

Among the fields of force whose trajectories are invariant under projective groups of three or more parameters, there are only eight projectively distinct types; these are given in the following table together with the largest group admitted.

No.	Group.	ϕ	ψ
6	q, yq, xq, p, xp	0	1
7	$p, q, xq, axp + yq \qquad a = 0$	0	e^x
10	xp, xq, yp, yq	x	y
12	q, xq, yq, xp	0	x^c
14	$p + xp, xp + 2yq, q$	1	x
18	$p, q, (a-1)xp + ayq \quad a = 0$	0	e^y
20	xp, yq, q	x^c	0
22	q, xq, yq	0	$F(x)$

The differential equations of the trajectories of these eight types, together with their complete solutions, are given below. The calculations require only quadratures.

Group [6]:

$$y''' = 0, \qquad y = \alpha x^2 + \beta x + \gamma.$$

Group [7]:

$$y''' = y'', \qquad y = \alpha e^x + \beta x + \gamma.$$

Group [10]:

$$y''' = \frac{-3xy''^2}{y - xy'}, \qquad \alpha x^2 + \beta xy + \gamma y^2 = 1.$$

Group [12]:

$$y''' = \frac{cy''}{x}, \qquad y = \alpha x^{c+2} + \beta x + \gamma.$$

Group [14]:

$$y''' = \frac{3y''^2}{y' - x} + \frac{y''}{x - y'},$$

$$4x^4 - 48x^2y - 16\alpha x^3 + 4x^2(\beta + 4\alpha^2) + 96\alpha xy + 144y^2 - 2x(4\alpha\beta + \gamma) - 24\beta y + \beta^2 - \alpha\gamma = 0.$$

Group [18]:

$$y''' = y'y'', \qquad \gamma e^y = \sec^2(\alpha x + \beta).$$

Group [20]:

$$y''' = \frac{3y''^2}{y'} + \frac{cy''}{x}, \qquad y + \gamma = \int \frac{dx}{\sqrt{\alpha x^{c+1} + \beta}}.$$

Group [22]:

$$y''' = \frac{F'(x)}{F(x)} y'', \qquad y = \alpha \int\int F(x)\,dx\,dx + \beta x + \gamma.$$

From these results we see that the trajectories invariant under groups [7] and [18] are transcendental curves; those invariant under [6], [10], [12] and [14] are algebraic; while those invariant under [20] and [22] may be either.

Of especial interest are those forces in which the trajectories are a triply infinite system of conics. This is obviously the case for the trajectories invariant under groups [6] and [10]. For the first, the system consists of parabolas with vertical axes; and for the second, of the conics with a common center.

When $c = -3$ or $-\frac{3}{2}$ the trajectories invariant under [12] are conics given by the equations

(1) $$xy = \beta x^2 + \gamma x + \alpha,$$

(2) $$(\alpha_1 y + \beta_1 x + \gamma_1)^2 = x.$$

Equation (1) is that of the system of hyperbolas whose centers are on the y-axis and which have the y-axis as a common asymptote, a system which may easily be projected into the system of vertical parabolas. The projection $x = (x_1 + iy_1)/(x_1 - iy_1)$, $y = 1/(x_1 - iy_1)$, which transforms the line at infinity and the y-axis into the two minimal lines through the origin, transforms (2), since that is the equation of all parabolas tangent to the y-axis, into the equation of the conics with the origin as a common focus. The force is then newtonian attraction.

When $c = -3$ the trajectories, under [20] have the equation

$$y^2 + \beta_1 x^2 + \alpha_1 y + \gamma_1 = 0.$$

This gives the system of conics with centers on the y-axis and axes parallel to the coördinates axes, projectively equivalent to the system under group [10]. For $c = 0$ the trajectories invariant under the group [22] obviously become the parabolas with parallel axes.

No new cases are found under group [22]. It is known from the investigations of BERTRAND, HALPHEN, and APPELL that there exist only three projectively distinct types where the trajectories are conic sections: *First*, parabolas with parallel axes, generated by a constant parallel force, and allowing a five-parameter group. *Second*, conics with a common center, generated by a central force varying directly as the distance, and allowing a four-parameter group. *Third*, conics with a common focus, generated by a central force varying inversely as the square of the distance, and allowing a four-parameter group.

COLUMBIA UNIVERSITY,
May, 1908.